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Optimal constant weight covering codes and nonuniform group divisible 3-designs with block size four

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Abstract Let $K_q(n, w, t, d)$ be the minimum size of a code over Z_q of length n, constant weight w, such that every word with weight t is within Hamming distance d of at least one codeword. In this article, we determine $K_q(n, 4, 3, 1)$ for all $n \geq 4$, $q = 3, 4$ or $q = 2^m + 1$ with $m \geq 2$, leaving the only case $(q, n) = (3, 5)$ in doubt. Our construction method is mainly based on the auxiliary designs, H-frames, which play a crucial role in the recursive constructions of group divisible 3-designs similar to that of candelabra systems in the constructions of 3-wise balanced designs. As an application of this approach, several new infinite classes of nonuniform group divisible 3-designs with block size four are also constructed.

Keywords Constant weight covering code · Group divisible t-covering · Group divisible t-design · H-frame

Mathematics Subject Classification (2000) Primary 05B40 · 94B65 · 05C35

1 Introduction

Let Z_q^n denote the set of all n-tuples over Z_q. The elements of Z_q^n are called *words*. For each word u and $i \in \{1, 2, \ldots, n\}$, u_i denotes the ith component of u. The *Hamming distance* between two words u, v is defined as $\Delta(u, v) = |\{i : u_i \neq v_i\}|$. The *Hamming weight* $wt(u)$ of u is defined as the distance from the origin, i.e., $wt(u) = \Delta(u, 0)$. An (n, w, t, d) *constant weight covering code* over Z_q is a subset of Z_q^n with constant weight w, such that every word with weight t is within Hamming distance d of at least one codeword. Denote the minimum size of such a code by $K_q(n, w, t, d)$ and a code achieving this size is called *optimal*.

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One of the main motivations for studying covering codes is their applications to universal data compression algorithms, see e.g. [3,8]. Consider a universal quantizer, which consists of a bank of optimal constant weight covering codes, one for each weight. An input vector x is compressed by the universal quantizer to a pair (i, j), where i indicates the selected code according to the weight of x and j indicates the codeword closest to x in the selected code. Apart from these applications, the determination of $K_q(n, w, t, d)$ is a fundamental combinatorial problem which was studied by many researchers in the last sixty years. In combinatorics, there are several kinds of equivalent combinatorial objects [4], such as Turán designs, lottery schemes and covering designs. For $w - t \geq 0$, an $(n, w, t, w - t)$ constant weight covering code over Z_q is equivalent to a kind of group divisible covering design. A great number of papers have been published on the lower and upper bounds of $K_q(n, w, t, d)$ for $q = 2$, see [2,5]. However only specific values of them are determined, such as $K_2(n, 2, t, t-2)$ [25], $K_2(n, 3, 2, 1)$ [6], $K_2(n, 4, 3, 1)$ [12], $K_2(n, w, 2, w-2)$ [14]. For $q \geq 3$, some results can be found in [13].

Group divisible t-designs have been studied for many years by numerous researchers for a variety of reasons, see e.g. [4]. Important applications of these designs include the construction of other types of combinatorial structures, e.g. pairwise balanced designs and frames. Recently, Keranen and Kreher [18] started the investigation on the existence of nonuniform group divisible 3-designs. However, the existence problem is rather difficult and the known results are far from complete despite the effort of several authors (see [18,27]).

In this article, we pay particular attention to the determination of $K_q(n, 4, 3, 1)$ for $q = 3, 4$ or $q = 2^m + 1$ with $m \geq 2$. The problem will be solved almost completely with the only case $(q, n) = (3, 5)$ undetermined. This result is obtained by solving an equivalent problem, that is, the existence of group divisible coverings of triples by quadruples using combinatorial tools. Besides these, we also focus on the existence problems for nonuniform group divisible 3-designs with block size four. Several new infinite classes of such designs will be constructed.

Our construction method for the above two problems is mainly based on the auxiliary designs, H-frames, which play a crucial role in the recursive constructions of group divisible 3-designs similar to that of candelabra systems in the constructions of 3-wise balanced designs. We believe that the theory of candelabra systems and H-frames will be proved useful for solving the general existence problems on both group divisible coverings of triples by quadruples and nonuniform group divisible 3-designs with block size four.

2 Preliminary

Let v and t be positive integers and K be a set of positive integers. A *group divisible t-covering* (or t-GDC) of order v with block sizes from K, denoted by GDC(t, K, v), is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

(1) X is a set of v elements (called *points*);
(2) $\mathcal{G} = \{G_1, G_2, \dots\}$ is a set of nonempty subsets (called *groups*) of X which partition X;
(3) $\mathcal{B}$ is a family of transverses (called *blocks*) of $\mathcal{G}$, each of cardinality from K, where a *transverse* is a subset of X that intersects any given group in at most one point;
(4) every t-element transverse of $\mathcal{G}$ is contained in at least one block.

The *excess* of a t-GDC is the multi-set of t-element transverses T of $\mathcal{G}$ with multiplicity $|\{B \in \mathcal{B} : T \subset B\}| - 1$. The *type* of the GDC$(t, K, v)$ is defined as the list $(|G| | G \in \mathcal{G})$. If a GDC has n_i groups of size g_i, $1 \leq i \leq r$, then we use an *"exponential"* notation

$g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r}$ to denote the group type. A t-GDC is called *uniform* if all of its groups have the same size. When $K = \{k\}$, we simply write k for K.

A t-GDC is known as *group divisible t-design* (or t-GDD), denoted by GDD(t, k, v), if every t-element transverse T of $\mathcal{G}$ is contained in exactly one block. Mills [22] used H$(n, g, 4, 3)$ design to denote the GDD$(3, 4, ng)$ of type g^n. A GDD(t, K, n) of type 1^n is also referred to as a *t-wise balanced design* of order n with block sizes from K, denoted by S(t, K, n). When $t = 3$ and $K = \{4\}$, it is well known as *Steiner quadruple systems* and denoted by SQS(n). Hanani [9] has shown that an SQS(n) exists for all $n \equiv 2, 4 \pmod 6$.

Theorem 2.1 ([16,22]) *For $n > 3$ and $n \neq 5$, a GDD$(3, 4, gn)$ of type g^n exists if and only if ng is even and $g(n-1)(n-2)$ is divisible by 3. For $n = 5$, a GDD$(3, 4, 5g)$ of type g^5 exists when g is even, $g \neq 2$ and $g \not\equiv 10, 26 \pmod{48}$.*

Let $C(n, g, k, t)$ denote the minimum number of blocks in any GDC(t, k, ng) of type g^n. A GDC(t, k, ng) of type g^n, $(X, \mathcal{G}, \mathcal{B})$, is *optimal* (OGDC) if $|\mathcal{B}| = C(n, g, k, t)$. Clearly, if a GDD$(t, k, ng)$ of type g^n exists, then it is optimal. Let $L(n, g, k, t) = \lceil \frac{gn}{k} \lceil \frac{g(n-1)}{k-1} \cdots \lceil \frac{g(n-t+1)}{k-t+1} \rceil \cdots \rceil \rceil$. Schönheim [23] showed that $C(n, g, k, t) \geq L(n, g, k, t)$ for all $n \geq k \geq t \geq 1$.

For $t = 3$, $k = 4$ and $g = 1$, Mills [20] has shown that $C(n, 1, 4, 3) = L(n, 1, 4, 3)$ for all $n \not\equiv 7 \pmod{12}$. Kalbfleisch and Stanton [17] and Swift [24] have shown that $C(7, 1, 4, 3) = L(7, 1, 4, 3) + 1 = 12$. Mills [21] also proved that $C(499, 1, 4, 3) = L(499, 1, 4, 3)$. Hartman et al. [12] have shown that $C(n, 1, 4, 3) = L(n, 1, 4, 3)$ for all $n \geq 52423$. It was recently improved by Ji [15] that $C(n, 1, 4, 3) = L(n, 1, 4, 3)$ for all n with an exception $n = 7$ and possible exceptions of $n = 12k + 7, k \in \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 16, 21, 23, 25, 29\}$.

Lemma 2.1 *The existence of an OGDC(t, k, ng) of type g^n is equivalent to that of an optimal $(n, k, t, k-t)$ constant weight covering code over Z_{g+1}, i.e., $C(n, g, k, t) = K_{g+1}(n, k, t, k-t)$.*

Proof Suppose we have an OGDC(t, k, ng) of type g^n, $(I_n \times I_g, \{\{i\} \times I_g : i \in I_n\}, \mathcal{B})$, where $I_s = \{1, 2, \ldots, s\}$. For each block $\{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\} \in \mathcal{B}$, we obtain a codeword of length n by putting b_j in the position of a_j, $1 \leq j \leq k$, and zeros elsewhere. It is easy to see that all the resultant codewords form an optimal $(n, k, t, k-t)$ constant weight covering code over Z_{g+1}.

Conversely, suppose we have an optimal $(n, k, t, k-t)$ constant weight covering code $\mathcal{C}$ over Z_{g+1}. For each codeword $u \in \mathcal{C}$, if the nonzero positions of u are $a_1, a_2, \ldots, a_k$, and the corresponding components of u are $b_1, b_2, \ldots, b_k$, then we form a block $\{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\}$. It is easy to check that all the resultant blocks form an OGDC(t, k, ng) on $I_n \times I_g$ with group set $\{\{i\} \times I_g : i \in I_n\}$. □

Now, we give the concept of an H-frame, which is a generalization of that in [12]. An *$H(t, K, v)$ frame* is an ordered four-tuple $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with the following properties:

1. X is a set of v points;
2. $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a partition of X into groups;
3. $\mathcal{F}$ is a family $\{F_i\}$ of subsets of $\mathcal{G}$ called *holes*, which is closed under intersections. Hence each hole $F_i \in \mathcal{F}$ is of the form $F_i = \{G_{i_1}, G_{i_2}, \ldots, G_{i_s}\}$, and if F_i and F_j are holes then $F_i \cap F_j$ is also a hole. The number of groups in a hole is its *size*; and
4. $\mathcal{B}$ is a set of transverses (called *blocks*) of $\mathcal{G}$, each of cardinality from K, such that every t-element transverse of $\mathcal{G}$ which is not a t-element transverse of any hole $F_i \in \mathcal{F}$ is

contained in precisely one block, and no block contains a t-element transverse of any hole.

If all the groups in $\mathcal{G}$ have the same size g, then $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ is an *$H(v/g, g, K, t)$ frame* as defined in [12], which is shortly denoted by HF$(v/g, g, K, t)$. If an HF$(q, g, K, 3)$ has n_i holes of size $m_i + s$ intersecting on a common hole of size s, $i = 1, 2, \ldots, r$, then we denote such a design as K-HF$_g(m_1^{n_1} m_2^{n_2} \cdots m_r^{n_r} : s)$. When $g = 1$, a K-HF$_1(m_1^{n_1} m_2^{n_2} \cdots m_r^{n_r} : s)$ is known as a *candelabra system*, denoted by K-CS$(m_1^{n_1} m_2^{n_2} \cdots m_r^{n_r} : s)$ $(X, S, \Gamma, \mathcal{B})$, where S is the common hole called *stem*, $\Gamma = \{F \setminus S : F \in \mathcal{F}\}$ is the set of *groups*. When $K = \{4\}$, it is called a *candelabra quadruple system* (CQS). If an HF$(q, g, K, 3)$ has only one hole of size s, then we call it an *incomplete group divisible design*, denoted by IGDD$((q : s), g, K, 3)$.

If an H$(3, K, v)$ frame, $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$, has the properties that all the groups in $\mathcal{G}$ have the same size g except for G_1, which has size $g - 1$ and $\mathcal{F}$ has n_i holes of size $m_i + s$ intersecting on a common hole of size s, $i = 1, 2, \ldots, r$, while G_1 belongs to the common hole, then we call such an H$(3, K, v)$ frame a *modified $H(3, K, v)$ frame*, which is denoted by K-MHF$_g(m_1^{n_1} m_2^{n_2} \cdots m_r^{n_r} : s)$.

Lemma 2.2 [20] *There exists a CQS$(6^n : 0)$ for all $n \geq 0$.*

Lemma 2.3 *For each integer $n \geq 3$, there exists a $\{4, 6\}$-CS$(2^n : 2)$.*

Proof For each $n \equiv 0, 1 \pmod 3$, $n \geq 3$, there exists a CQS$(2^n : 2)$ obtained from an SQS$(2n+2)$. For each $n \equiv 2 \pmod 3$, $n \geq 5$, there exists a $\{4, 6\}$-CS$(2^n : 2)$ obtained from a CQS$(6^{(n+1)/3} : 0)$ by taking two points from two distinct groups as stem points. □

Lemma 2.4 *Suppose that $(X, S, \Gamma, \mathcal{A})$ is a K-CS$(m^n : s)$ and $\infty \in S$. Let $K_1 = \{|A| : \infty \in A \in \mathcal{A}\}$ and $K_2 = \{|A| : \infty \notin A \in \mathcal{A}\}$. If there exist a 4-HF$_g(t^{k_1-1} : a)$(4-MHF$_g(t^{k_1-1} : a)$) for each $k_1 \in K_1$ and a GDD$(3, 4, gtk_2)$ of type $(gt)^{k_2}$ for each $k_2 \in K_2$, then there exists a 4-HF$_g((tm)^n : t(s-1) + a)$ (4-MHF$_g((tm)^n : t(s-1) + a)$ respectively).*

Proof Suppose that the given K-CS$(m^n : s)$ has group set $\Gamma = \{G_1, \ldots, G_n\}$. We first construct a 4-HF$_g((tm)^n : t(s-1) + a)$. Define $G'_{x,j} = x \times \{j\} \times Z_g$. Let $X' = ((X \setminus \{\infty\}) \times Z_t \times Z_g) \cup (\{\infty\} \times Z_a \times Z_g)$, $\mathcal{G}' = \{G'_{x,j} : x \in X \setminus \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$, $\mathcal{F} = \{F_i : 0 \leq i \leq n\}$, where $F_0 = \{G'_{x,j} : x \in S \setminus \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$ being the common hole of size $t(s-1) + a$ and $F_i = \{G'_{x,j} : x \in G_i, j \in Z_t\} \cup F_0$ for $1 \leq i \leq n$.

For each $B \in \mathcal{A}$ and $\infty \in B$, construct a 4-HF$_g(t^{|B|-1} : a)$ on $((B \setminus \{\infty\}) \times Z_t \times Z_g) \cup (\{\infty\} \times Z_a \times Z_g)$ with group set $\{G'_{x,j} : x \in B \setminus \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$ and holes $F_x = \{G'_{x,j} : j \in Z_t\} \cup F_\infty$, $x \in B \setminus \{\infty\}$ intersecting on a common hole $F_\infty = \{G'_{\infty,j} : j \in Z_a\}$ of size a. Denote its block set by $\mathcal{A}_B$.

For each $B \in \mathcal{A}$ and $\infty \notin B$, construct a GDD$(3, 4, gt|B|)$ of type $(gt)^{|B|}$ on $B \times Z_t \times Z_g$ with group set $\{x \times Z_t \times Z_g : x \in B\}$. Denote its block set by $\mathcal{C}_B$.

Let $\mathcal{A}' = (\cup_{B \in \mathcal{A}, \infty \in B} \mathcal{A}_B) \cup (\cup_{B \in \mathcal{A}, \infty \notin B} \mathcal{C}_B)$. It is easy to check that $\mathcal{A}'$ is the block set of a 4-HF$_g((tm)^n : t(s-1) + a)$ on X' with group set $\mathcal{G}'$ and hole set $\mathcal{F}$.

The proof for the construction of a 4-MHF$_g((tm)^n : t(s-1) + a)$ is similar as above. Denote $Z_a^* = Z_a \setminus \{0\}$. Let $X'' = ((X \setminus \{\infty\}) \times Z_t \times Z_g) \cup (\{\infty\} \times ((Z_a \times Z_g) \setminus \{(0, 0)\}))$, $\mathcal{G}'' = \{G'_{x,j} : x \in X \setminus \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a^*\} \cup \{G'_{\infty,0} \setminus \{(\infty, 0, 0)\}\}$, $\mathcal{F}' = \{F'_i : 0 \leq i \leq n\}$, where $F'_0 = \{G'_{x,j} : x \in S \setminus \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a^*\} \cup \{G'_{\infty,0} \setminus \{(\infty, 0, 0)\}\}$ being the common hole of size $t(s-1) + a$ and $F'_i = \{G'_{x,j} : x \in G_i, j \in Z_t\} \cup F'_0$ for $1 \leq i \leq n$. It is easy to get a 4-MHF$_g((tm)^n : t(s-1) + a)$ on X'' with group set $\mathcal{G}''$ and hole set $\mathcal{F}'$ by taking the above similar steps as those for a 4-HF$_g((tm)^n : t(s-1) + a)$. □

3 Optimal ternary constant weight covering codes

In this section, we determine $K_3(n, 4, 3, 1)$, i.e., $C(n, 2, 4, 3)$. From Theorem 2.1, there exists a GDD$(3, 4, 2n)$ of type 2^n if $n \equiv 1, 2 \pmod 3$ and $n \neq 5$, which means $C(n, 2, 4, 3) = L(n, 2, 4, 3)$ for all such n. For $n = 5$, we give a lower and an upper bound for $C(5, 2, 4, 3)$.

Lemma 3.1 $L(5, 2, 4, 3) + 2 \leq C(5, 2, 4, 3) \leq 24$.

Proof It is easy to construct a GDC$(3, 4, 10)$ of type 2^5 with 24 blocks. Let $X = Z_8$ and $\mathcal{G} = \{\{i, i+4\} : i = 0, 1, 2, 3\}$. There exist a GDD$(3, 4, 8)$ and a GDD$(2, 3, 8)$ of type 2^4 on X with group set $\mathcal{G}$. Denote the block sets by $\mathcal{B}$ and $\mathcal{T}$ respectively, both of which have cardinality 8. Let $X' = X \cup \{\infty_1, \infty_2\}$ and $\mathcal{G}' = \mathcal{G} \cup \{\{\infty_1, \infty_2\}\}$. For each $i = 1, 2$, let $\mathcal{C}_i = \{T \cup \{\infty_i\} : T \in \mathcal{T}\}$. Then it is easy to show that $\mathcal{B} \cup \mathcal{C}_1 \cup \mathcal{C}_2$ is a GDC$(3, 4, 10)$ of type 2^5 on X' with group set $\mathcal{G}'$ having 24 blocks. Hence, $C(5, 2, 4, 3) \leq 24$.

For the lower bound, we only need to prove the nonexistence of a GDC$(3, 4, 10)$ of type 2^5 with 21 blocks. Assume $(X', \mathcal{G}', \mathcal{A})$ is a GDC$(3, 4, 10)$ of type 2^5 with $|\mathcal{A}| = 21$ and excess E. Then $|E| = 4$ and E contains 12 elements with at least three of them being distinct. Suppose E contains five or more distinct points. Then, E contains at least 15 elements noting the fact that: for each point appears in the excess, the number of triples in the excess containing this point is divisible by 3. Suppose E contains exactly three distinct points, say a, b, c. Then, the repetition numbers for them to appear in E could only be 3, 3, 6. There is no way to form four triples on $\{a, b, c\}$. Thus E must contain four distinct points, say a, b, c, d, each of which is contained in three triples of E. So E should be composed of $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. Furthermore, $\{a, b, c, d\} \notin \mathcal{A}$. Otherwise, we can delete $\{a, b, c, d\}$ from the block set to get a GDD$(3, 4, 10)$ of type 2^5. This contradicts to the nonexistence of such a design. Hence, we conclude that $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ are contained in exactly eight blocks of $\mathcal{A}$. Since any two triples in $\{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ intersect in two common points and $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ are the only four triples in the excess, the fourth points in the eight blocks of $\mathcal{A}$ should be pairwise distinct. Otherwise, it will produce more triples in the excess. Consequently, we will have at least 12 distinct points in X'. This contradicts to the fact $|X'| = 10$. □

When $n \equiv 0 \pmod 3$, $L(n, 2, 4, 3) = n(n^2 - 3n + 3)/3$. We say a GDD$(3, \{4, 6\}, 2n)$ of type 2^n is *good* if it contains exactly $n/3$ blocks of size 6.

Lemma 3.2 *If there exists a good GDD$(3, \{4, 6\}, 2n)$ of type 2^n, then $C(n, 2, 4, 3) = L(n, 2, 4, 3)$.*

Proof The number of blocks of size 4 for a good GDD$(3, \{4, 6\}, 2n)$ of type 2^n is $n(n^2 - 3n - 3)/3$. For each block of size 6, replace it with an OGDC$(3, 4, 6)$ of type 1^6. Since $C(6, 1, 4, 3) = 6$, the conclusion then follows. □

Note that a GDD$(3, \{4, 6\}, 2n)$ of type 2^n was constructed for all $n \equiv 0 \pmod 3$ in [26], but not all of them are good.

Lemma 3.3 [26] *There exists a good GDD$(3, \{4, 6\}, 2n)$ of type 2^n for $n = 6, 9, 15$.*

Lemma 3.4 *There exists a good GDD$(3, \{4, 6\}, 2n)$ of type 2^n for all $n \equiv 0 \pmod 6$.*

Proof For each $n = 6k$ with $k \geq 1$, take a CQS$(6^k : 0)$, which exists by Lemma 2.2. Give each point weight two, and construct a GDD$(3, 4, 8)$ of type 2^4 for each block to get a 4-HF$_2(6^k : 0)$. For each hole of the 4-HF$_2(6^k : 0)$, fill a good GDD$(3, \{4, 6\}, 12)$ of type 2^6 to get a good GDD$(3, \{4, 6\}, 12k)$ of type 2^{6k} as desired. □

Lemma 3.5 *There exists an* IGDD$((9:3),2,\{4,6\},3)$ *with exactly two blocks of size* 6.

Proof The IGDD$((9 : 3), 2, \{4, 6\}, 3)$ is constructed on Z_{18} with nine groups $\{i, i+9\}$, $i = 0, 1, \ldots, 8$ and one hole $\{\{i, i+9\} : i = 6, 7, 8\}$. The two blocks of size 6 are $\{0, 1, 2, 3, 4, 5\}$ and $\{9, 10, 11, 12, 13, 14\}$. The remaining 52×3 blocks of size 4 are obtained by developing the following 52 blocks under the automorphism group $\langle(0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11)(12\ 13\ 14)(15\ 16\ 17)\rangle$.

$$\begin{array}{lllll}
\{3,5,6,7\} & \{5,6,11,17\} & \{0,6,12,17\} & \{0,5,10,17\} & \{1,3,9,13\}\\
\{0,3,13,16\} & \{1,4,9,15\} & \{6,7,9,11\} & \{0,4,10,12\} & \{3,6,13,14\}\\
\{0,11,14,15\} & \{2,6,14,17\} & \{0,1,6,7\} & \{1,2,13,14\} & \{0,10,14,16\}\\
\{5,8,11,16\} & \{2,12,13,16\} & \{3,5,9,11\} & \{3,7,11,13\} & \{0,1,8,14\}\\
\{1,2,15,16\} & \{0,4,15,16\} & \{0,2,10,15\} & \{2,7,9,12\} & \{6,8,12,14\}\\
\{2,4,8,15\} & \{6,8,9,13\} & \{0,5,13,15\} & \{5,9,13,16\} & \{3,4,14,15\}\\
\{4,9,11,17\} & \{5,7,11,15\} & \{1,4,7,11\} & \{1,4,8,12\} & \{6,11,12,16\}\\
\{6,10,13,16\} & \{0,4,7,17\} & \{1,8,9,11\} & \{4,6,9,12\} & \{9,11,15,16\}\\
\{3,4,16,17\} & \{11,13,16,17\} & \{12,13,15,17\} & \{2,7,14,15\} & \{6,9,14,16\}\\
\{0,3,8,15\} & \{2,4,6,7\} & \{0,4,6,14\} & \{0,4,8,11\} & \{0,6,11,13\}\\
\{3,4,6,11\} & \{3,10,13,15\}. & & &
\end{array}$$

□

Lemma 3.6 *There exists a* 4-$HF_2(3^5 : 0)$.

Proof The design is constructed on Z_{30} with groups $G_i = \{i, i+15\}$, $i = 0, 1, \ldots, 14$ and five holes $F_i = \{G_i, G_{i+5}, G_{i+10}\}$, $i = 0, 1, \ldots, 4$. The following blocks are developed under the automorphism group Z_{30}.

$$\begin{array}{lllll}
\{0,6,13,27\} & \{0,8,11,29\} & \{0,3,12,17\} & \{0,4,10,23\} & \{0,1,4,7\}\\
\{0,19,24,28\} & \{0,2,7,16\} & \{0,1,20,26\} & \{0,7,14,26\} & \{0,18,23,26\}\\
\{0,4,26,29\} & \{0,6,7,12\} & \{0,13,17,29\} & \{0,11,12,19\} & \{0,16,18,22\}\\
\{0,3,20,23\} & \{0,1,3,28\} & \{0,5,7,13\} & \{0,5,17,26\} & \{0,2,4,13\}\\
\{0,13,19,22\} & \{0,9,10,19\} & \{0,5,6,22\} & \{0,10,14,24\} & \{0,5,16,19\}\\
\{0,1,11,13\} & \{0,2,9,22\} & \{0,2,10,18\} & \{0,2,12,24\} & \{0,21,28,29\}.
\end{array}$$

□

The following result is based on the construction of a CQS$((6n)^3 : 2s)$ by Hartman in [11, Sect. 4], where the major ingredients in the construction are a class of auxiliary designs called *A-pairings*.

Theorem 3.1 *There exists a* 4-$HF_2((3n)^3 : s)$ *for all* $3n \geq s \geq 0$.

Proof For all $3n \geq s \geq 0$ and $(n, s) \neq (1, 1)$, Hartman in [11, Sect. 4] constructed a CQS$((6n)^3 : 2s)$ on $X = \{a_i : a \in Z_{6n}, i \in Z_3\} \cup \{\infty_1, \infty_2, \ldots, \infty_{2s}\}$ with three groups $\{\{a_i : a \in Z_{6n}\} : i \in Z_3\}$ and stem $\{\infty_1, \infty_2, \ldots, \infty_{2s}\}$. Let the block set be $\mathcal{B}$, in which there is a set of blocks

$$\begin{aligned}\phi = \{\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a, b\} \in F_i^{(k)}, \{c, d\} \in F_{i+1}^{(k)},\\ 1 \leq k \leq 6n - 1 - 2r - 2h, i \in Z_3\},\end{aligned}$$

where $F_i^{(1)}|F_i^{(2)}|\ldots|F_i^{(6n-1-2r-2h)}$ is a one-factorization of the graph on $Z_{6n} \times \{i\}$ defined by the A-pairing $A(n, 2s)$. By the detailed construction of $A(n, 2s)$ in [11, Sect. 5], we know that $6n - 1 - 2r - 2h \geq 1$.

The desired 4-$HF_2((3n)^3 : s)$ will be on X with the group set $\mathcal{G} = \{\{a_i, b_i\} : \{a, b\} \in F_i^{(1)}, i \in Z_3\} \cup \{\{\infty_i, \infty_{i+s}\} : 1 \le i \le s\}$, three holes $\mathcal{F}_{i+1} = \{\{a_i, b_i\} : \{a, b\} \in F_i^{(1)}\} \cup \mathcal{F}_0, i \in Z_3$ intersecting on a common hole $\mathcal{F}_0 = \{\{\infty_i, \infty_{i+s}\} : 1 \le i \le s\}$.

Let

$$\phi_1 = \{\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a, b\} \in F_i^{(1)}, \{c, d\} \in F_{i+1}^{(1)}, i \in Z_3\}.$$

Note that $\phi_1 \subset \phi$ and each block in ϕ_1 intersects two groups in $\mathcal{G}$ which are from two distinct holes. It is readily checked that $\mathcal{B} \setminus \phi_1$ is the block set of the desired 4-$HF_2((3n)^3 : s)$.

For $(n, s) = (1, 1)$, a 4-$HF_2(3^3 : 1)$ can be obtained by applying Lemma 2.4 with a CQS$(3^3 : 1)$ in [9] and a GDD$(3, 4, 8)$ of type 2^4. □

Lemma 3.7 *There exists a good GDD$(3, \{4, 6\}, 2n)$ of type 2^n for all $n \equiv 3 \pmod 6$ and $n \ge 9$.*

Proof For $n = 9, 15$, the designs come from Lemma 3.3. For each $n = 6m + 3$ with $m \ge 3$, there exists a $\{4, 6\}$-CS$(2^m : 2)$ by Lemma 2.3. Apply Lemma 2.4 with $g = 2$, $t = 3$ and $a = 0$ to get a 4-$HF_2(6^m : 3)$. Here the input 4-$HF_2(3^{k-1} : 0)$s and GDD$(3, 4, 6k)$s of type 6^k with $k \in \{4, 6\}$ exist by Theorems 2.1, 3.1 and Lemma 3.6. For the first $m - 1$ holes of the 4-$HF_2(6^m : 3)$, fill the IGDD$((9 : 3), 2, \{4, 6\}, 3)$ constructed in Lemma 3.5. Fill the last hole with a good GDD$(3, \{4, 6\}, 18)$ of type 2^9. The result is a good GDD$(3, \{4, 6\}, 2n)$ of type 2^n. □

Combining Lemmas 3.1, 3.2, 3.4 and 3.7, we have

Theorem 3.2 $C(n, 2, 4, 3) = L(n, 2, 4, 3)$ *for all $n \ge 4$ and $n \ne 5$.*

4 Optimal quaternary constant weight covering codes

In this section, we determine $K_4(n, 4, 3, 1)$, i.e., $C(n, 3, 4, 3)$. From Theorem 2.1, there exists a GDD$(3, 4, 3n)$ of type 3^n if $n \equiv 0 \pmod 2$, i.e., $C(n, 3, 4, 3) = L(n, 3, 4, 3)$ for all such n. When $n \equiv 1 \pmod 2$, $L(n, 3, 4, 3) = 3n(n - 1)(3n - 5)/8$. A GDC$(3, 4, 3n)$ of type 3^n is called *good* if the excess forms a GDD$(2, 3, 3n)$ of type 3^n.

Lemma 4.1 *If there exists a good GDC$(3, 4, 3n)$ of type 3^n, then $C(n, 3, 4, 3) = L(n, 3, 4, 3)$.*

Proof It is easy to check that the number of blocks in a good GDC$(3, 4, 3n)$ of type 3^n equals $L(n, 3, 4, 3)$. □

Lemma 4.2 *There exists a good GDC$(3, 4, 3n)$ of type 3^n for each $n \in \{5, 7, 9, 11\}$.*

Proof For each given $n \in \{5, 7, 9, 11\}$, a good GDC$(3, 4, 3n)$ of type 3^n is constructed on Z_{3n} with groups $\{i, i + n, i + 2n\}$, $i = 0, 1, \ldots, n - 1$. The following blocks are developed under the automorphism group Z_{3n}.

$n = 5:$

$\{0, 1, 2, 4\}$ $\{0, 1, 4, 13\}$ $\{0, 1, 7, 9\}$ $\{0, 1, 8, 12\}$ $\{0, 2, 8, 11\}$;

$n = 7:$

$\{0, 11, 15, 17\}$ $\{0, 12, 18, 20\}$ $\{0, 8, 9, 11\}$ $\{0, 1, 6, 11\}$ $\{0, 16, 17, 20\}$
$\{0, 2, 8, 18\}$ $\{0, 3, 5, 8\}$ $\{0, 9, 12, 15\}$ $\{0, 8, 19, 20\}$ $\{0, 4, 8, 16\}$
$\{0, 1, 12, 16\}$ $\{0, 2, 4, 13\}$;

$n = 9:$

$\{0, 5, 21, 25\}$ $\{0, 6, 13, 17\}$ $\{0, 10, 11, 22\}$ $\{0, 4, 7, 17\}$ $\{0, 21, 23, 26\}$
$\{0, 2, 12, 13\}$ $\{0, 3, 15, 19\}$ $\{0, 12, 22, 25\}$ $\{0, 13, 24, 26\}$ $\{0, 12, 19, 20\}$
$\{0, 5, 19, 24\}$ $\{0, 1, 3, 11\}$ $\{0, 4, 12, 23\}$ $\{0, 5, 7, 20\}$ $\{0, 3, 25, 26\}$
$\{0, 5, 22, 26\}$ $\{0, 5, 11, 12\}$ $\{0, 6, 8, 14\}$ $\{0, 3, 6, 10\}$ $\{0, 6, 13, 16\}$
$\{0, 1, 7, 13\}$ $\{0, 2, 4, 19\}$;

$n = 11:$

$\{0, 1, 10, 14\}$ $\{0, 17, 18, 21\}$ $\{0, 21, 23, 29\}$ $\{0, 25, 29, 32\}$ $\{0, 9, 19, 25\}$
$\{0, 9, 14, 26\}$ $\{0, 1, 13, 17\}$ $\{0, 8, 16, 23\}$ $\{0, 2, 6, 19\}$ $\{0, 17, 20, 30\}$
$\{0, 1, 15, 16\}$ $\{0, 21, 28, 31\}$ $\{0, 2, 5, 14\}$ $\{0, 4, 6, 9\}$ $\{0, 3, 8, 24\}$
$\{0, 8, 10, 29\}$ $\{0, 13, 21, 26\}$ $\{0, 7, 9, 24\}$ $\{0, 5, 6, 8\}$ $\{0, 4, 12, 13\}$
$\{0, 14, 17, 23\}$ $\{0, 2, 10, 28\}$ $\{0, 1, 5, 31\}$ $\{0, 13, 15, 28\}$ $\{0, 2, 16, 31\}$
$\{0, 9, 15, 21\}$ $\{0, 8, 18, 28\}$ $\{0, 6, 13, 25\}$ $\{0, 15, 27, 30\}$ $\{0, 1, 8, 32\}$
$\{0, 23, 28, 32\}$ $\{0, 7, 14, 30\}$ $\{0, 20, 26, 27\}$ $\{0, 1, 6, 20\}$ $\{0, 4, 10, 19\}$.

□

To introduce the recursive construction, we need to define the concept of incomplete good group divisible coverings. Let X be the point set of size $3n$, where $\mathcal{G}$ is the equipartition of X into n groups of size 3. Suppose $\mathcal{H} \subset \mathcal{G}$ is a hole of size m, which is a collection of m groups. $\mathcal{B}$ is a family of transverse quadruples (*blocks*) such that, no block contains transverse triples in the hole, each transverse triple not in the hole occurs at least once in the blocks, and the excess forms a GDD(2, 3, $3n$) of type $3^{n-m}(3m)^1$. Then $(X, \mathcal{G}, \mathcal{H}, \mathcal{B})$ is called an *incomplete* good GDC(3, 4, $3n$) of type $(3^n : 3^m)$.

Lemma 4.3 *Suppose that there exists a* 4-*MHF*$_3(m^n : s)$. *If there exists an incomplete good GDC*$(3, 4, 3(m+s))$ *of type* $(3^{m+s} : 3^s)$*, then there exist incomplete good GDC*$(3, 4, 3(mn+s))$*s of types* $(3^{mn+s} : 3^s)$ *and* $(3^{mn+s} : 3^{m+s})$*. Furthermore, if there exists a good GDC*$(3, 4, 3(m+s))$ *of type* 3^{m+s}*, then there exists a good GDC*$(3, 4, 3(mn+s))$ *of type* 3^{mn+s}.

Proof Let $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ be the given 4-MHF$_3(m^n : s)$, where $G_1 = \{\alpha, \beta\}$ is the special group of size 2 belonging to the common hole F_0. Let $G_1' = G_1 \cup \{\infty\}$, where $\infty \notin X$. Let $X' = X \cup \{\infty\}$, $\mathcal{G}' = \mathcal{G} \cup \{G_1'\} \setminus \{G_1\}$, $\mathcal{F}' = \{F \cup \{G_1'\} \setminus \{G_1\} : F \in \mathcal{F}\}$. Let $T_\alpha = \{B \setminus \{\alpha\} : \alpha \in B \in \mathcal{B}\}$. Then T_α forms a GDD$(2, 3, 3mn)$ of type $(3m)^n$ on $X \setminus (\cup_{G \in F_0} G)$ with group set $\{\cup_{G \in (F \setminus F_0)} G : F \in \mathcal{F}, F \neq F_0\}$. Let $B_\infty = \{T \cup \{\infty\} : T \in T_\alpha\}$ and $\mathcal{B}' = \mathcal{B} \cup B_\infty$. For each hole $F' \in \mathcal{F}'$ of size $m+s$, construct an incomplete good GDC$(3, 4, 3(m+s))$ of type $(3^{m+s} : 3^s)$ with a hole $F_0' = F_0 \cup \{G_1'\} \setminus \{G_1\}$. Denote the block set by $\mathcal{A}_{F'}$, the excess of which forms a GDD$(2, 3, 3(m+s))$ of type $3^m(3s)^1$ with the long group $\cup_{G \in F_0'} G$. Let $\mathcal{C} = \mathcal{B}' \cup (\cup_{F' \in \mathcal{F}', F' \neq F_0'} \mathcal{A}_{F'})$. It is easy to check that the excess of $\mathcal{C}$ forms a GDD$(2, 3, 3(mn+s))$ of type $3^{mn}(3s)^1$ with the long group $\cup_{G \in F_0'} G$. Hence $\mathcal{C}$ is the block set of an incomplete good GDC$(3, 4, 3(mn+s))$ of type $(3^{mn+s} : 3^s)$ on X' with group set $\mathcal{G}'$ and a hole F_0'. If we leave the last hole as it is, or fill it with a good GDC$(3, 4, 3(m+s))$

of type 3^{m+s}, then we get an incomplete good GDC$(3, 4, 3(mn+s))$ of type $(3^{mn+s} : 3^{m+s})$ or a good GDC$(3, 4, 3(mn+s))$ of type 3^{mn+s}. □

Lemma 4.4 *There exists an incomplete good GDC*$(3, 4, 21)$ *of type* $(3^7 : 3^3)$.

Proof The design is constructed on Z_{21} with seven groups $\{i, i+7, i+14\}$, $i = 0, 1, \ldots, 6$ and one hole $\{\{i, i+7, i+14\} : i = 4, 5, 6\}$. The following blocks are developed by the automorphism group

$$\langle(0\ 7\ 14)(1\ 2\ 3\ 8\ 9\ 10\ 15\ 16\ 17)(4\ 5\ 6\ 11\ 12\ 13\ 18\ 19\ 20)\rangle.$$

{1, 7, 13, 18}	{11, 1, 7, 12}	{18, 8, 9, 12}	{0, 8, 10, 13}	{0, 4, 6, 16}
{0, 10, 11, 16}	{16, 4, 8, 14}	{6, 14, 17, 19}	{2, 5, 15, 18}	{9, 0, 6, 8}
{2, 3, 5, 20}	{0, 3, 8, 11}	{1, 10, 12, 16}	{6, 7, 8, 19}	{1, 5, 13, 17}
{6, 8, 11, 14}	{6, 7, 12, 16}	{10, 13, 16, 18}	{3, 13, 15, 19}	{8, 10, 19, 20}
{7, 11, 13, 16}	{2, 6, 7, 17}	{6, 2, 10, 11}	{0, 2, 6, 19}	{1, 2, 12, 17}
{1, 6, 9, 19}	{0, 2, 3, 15}.			

□

Lemma 4.5 *There exists an incomplete good GDC*$(3, 4, 27)$ *of type* $(3^9 : 3^3)$.

Proof The design is constructed on Z_{27} with groups $\{i, i+9, i+18\}$, $i = 0, 1, \ldots, 8$ and one hole $\{\{i, i+9, i+18\} : i = 6, 7, 8\}$. The following blocks are developed by the automorphism group

$$\langle(0\ 1\ 2\ 9\ 10\ 11\ 18\ 19\ 20)(3\ 4\ 5\ 12\ 13\ 14\ 21\ 22\ 23)(6\ 7\ 8)(15\ 16\ 17)(24\ 25\ 26)\rangle.$$

{5, 7, 18, 19}	{10, 4, 11, 26}	{5, 9, 12, 24}	{6, 11, 19, 25}	{4, 11, 23, 25}
{2, 5, 8, 15}	{0, 11, 25, 26}	{10, 11, 12, 23}	{8, 11, 22, 24}	{0, 8, 13, 16}
{1, 8, 15, 18}	{7, 13, 21, 23}	{8, 14, 18, 25}	{4, 7, 8, 19}	{0, 5, 13, 15}
{22, 9, 20, 21}	{7, 19, 21, 24}	{10, 18, 20, 22}	{22, 14, 16, 18}	{5, 15, 18, 20}
{3, 9, 13, 26}	{5, 7, 13, 26}	{4, 7, 20, 21}	{2, 3, 16, 22}	{7, 11, 14, 18}
{9, 16, 17, 19}	{0, 5, 7, 24}	{8, 0, 2, 7}	{11, 12, 17, 24}	{10, 18, 23, 26}
{3, 9, 20, 24}	{1, 2, 13, 16}	{10, 11, 14, 25}	{2, 13, 18, 19}	{2, 7, 14, 19}
{14, 1, 22, 24}	{3, 4, 6, 10}	{13, 14, 15, 17}	{2, 3, 6, 23}	{1, 17, 23, 24}
{5, 16, 18, 24}	{4, 17, 18, 24}	{3, 6, 17, 20}	{0, 3, 4, 11}	{11, 21, 22, 23}
{11, 14, 19, 22}	{1, 12, 17, 22}	{25, 0, 17, 20}	{0, 2, 22, 24}	{8, 3, 6, 13}
{10, 22, 25, 26}	{3, 7, 17, 23}	{4, 5, 24, 25}	{2, 4, 14, 16}	{5, 15, 21, 26}
{0, 3, 7, 10}	{5, 8, 11, 19}	{0, 4, 8, 20}	{8, 11, 16, 21}	{22, 2, 5, 12}
{4, 1, 14, 26}	{13, 16, 19, 21}	{0, 2, 16, 23}	{0, 3, 15, 17}	{0, 15, 19, 20}.

□

Lemma 4.6 *There exists a* 4*-MHF*$_3(2^3 : 1)$.

Proof The design is constructed on Z_{20} with groups $G_i = \{i, i+6, i+12\}$, $i = 0, 1, \ldots, 5$, and three holes $\{G_i, G_{i+3}\} \cup S$, $i = 0, 1, 2$ intersecting on the common hole $S = \{\{18, 19\}\}$. The desired blocks are obtained by developing the following base blocks under the automorphism group

$\langle(0\ 6\ 12)(1\ 7\ 13)(2\ 8\ 14)(3\ 9\ 15)(4\ 10\ 16)(5\ 11\ 17)(18)(19)\rangle$.

{3, 5, 6, 8}	{0, 10, 11, 18}	{2, 10, 13, 17}	{0, 2, 9, 10}	{4, 8, 12, 13}
{0, 8, 16, 18}	{0, 3, 4, 7}	{2, 3, 10, 19}	{0, 14, 16, 19}	{9, 10, 11, 19}
{6, 9, 11, 16}	{0, 7, 9, 11}	{1, 9, 17, 18}	{4, 9, 17, 19}	{6, 11, 13, 19}
{0, 7, 15, 16}	{0, 1, 4, 15}	{5, 8, 10, 13}	{0, 5, 13, 16}	{1, 5, 9, 19}
{0, 2, 4, 5}	{2, 9, 16, 18}	{7, 10, 14, 15}	{1, 11, 15, 18}	{4, 5, 6, 9}
{5, 10, 12, 14}	{0, 4, 11, 13}	{6, 7, 17, 19}	{0, 4, 17, 18}	{1, 2, 4, 12}
{1, 2, 5, 10}	{4, 7, 9, 12}	{1, 8, 10, 15}	{2, 3, 5, 7}	{2, 5, 9, 12}
{3, 8, 12, 16}	{12, 13, 14, 15}	{5, 9, 14, 16}	{1, 8, 11, 12}	{2, 5, 15, 16}
{0, 10, 17, 19}	{3, 7, 8, 19}	{0, 9, 13, 17}	{0, 1, 8, 17}	{2, 7, 9, 19}
{5, 8, 12, 15}	{0, 4, 8, 19}	{10, 11, 13, 14}	{1, 2, 9, 11}	{11, 13, 15, 16}
{2, 3, 12, 17}	{0, 3, 8, 13}	{4, 5, 13, 15}	{1, 4, 5, 14}	{10, 15, 17, 18}
{3, 10, 12, 13}	{2, 3, 6, 11}	{12, 13, 17, 18}	{0, 2, 13, 19}	{0, 2, 7, 18}
{0, 13, 14, 18}	{1, 3, 8, 18}	{2, 3, 4, 18}.		

□

Lemma 4.7 *There exists a* 4-*MHF*$_3(2^5 : 1)$.

Proof The design is constructed on Z_{32} with groups $G_i = \{i, i+10, i+20\}$, $i = 0, 1, \ldots, 9$ and five holes $\{G_i, G_{i+5}\} \cup S$, $i = 0, 1, 2, 3, 4$ intersecting on the common hole $S = \{\{30, 31\}\}$. The desired blocks are obtained by developing the following base blocks under the automorphism group

$\langle(0\ 1\ 2\ 3\ 4\ 5\ 6\ \cdots 21\ 22\ 23\ 24\ 25\ 26\ 27\ 28\ 29)(30\ 31)\rangle$.

{12, 17, 26, 28}	{15, 24, 27, 31}	{10, 18, 22, 24}	{2, 6, 17, 19}	{6, 12, 19, 31}
{7, 19, 22, 25}	{1, 9, 23, 27}	{2, 7, 19, 24}	{6, 7, 23, 24}	{3, 6, 7, 22}
{0, 4, 18, 31}	{2, 3, 5, 10}	{18, 25, 26, 29}	{2, 11, 18, 20}	{6, 14, 23, 31}
{1, 18, 22, 27}	{2, 3, 4, 31}	{3, 18, 24, 25}	{6, 11, 14, 27}	{13, 15, 19, 22}
{19, 22, 24, 26}	{4, 23, 25, 28}	{12, 15, 26, 30}	{18, 19, 23, 24}	{2, 6, 13, 30}
{7, 12, 19, 26}	{11, 13, 19, 31}	{2, 13, 19, 21}	{12, 14, 15, 29}	{7, 8, 26, 29}
{0, 1, 7, 12}	{0, 6, 14, 21}	{2, 9, 15, 21}.		

□

Lemma 4.8 *There exists a good GDC*$(3, 4, 3n)$ *of type* 3^n *for all* $n \equiv 3 \pmod 4$ *and* $n \geq 7$.

Proof For $n = 7, 11$, the required designs come from Lemma 4.2. For each $n = 4m+3$, $m \geq 3$, take a $\{4, 6\}$-CS$(2^m : 2)$ in Lemma 2.3. Applying Lemma 2.4 with a 4-MHF$_3(2^{k-1} : 1)$ and a GDD$(3, 4, 6k)$ of type 6^k, $k \in \{4, 6\}$, we get a 4-MHF$_3(4^m : 3)$. Then apply Lemma 4.3 with $m-1$ incomplete good GDC$(3, 4, 21)$s of type $(3^7 : 3^3)$ and a good GDC$(3, 4, 21)$ of type 3^7 to complete the proof. Here the ingredient designs come from Lemmas 4.4, 4.6 and 4.7. □

The following result on a 4-MHF$_3((2n)^3 : s)$ is again based on the construction of CQS$((6n)^3 : 2s)$ by Hartman in [11, Sect. 4]. First we introduce some useful notations. For $x \in Z_n$, we define $|x|$ by x if $0 \leq x \leq n/2$ and $n - x$ if $n/2 < x < n$. For any edge set E of a graph on Z_n, we use LE to denote the set of edge *lengths* of edges in E, i.e., $LE = \{|x - y| : \{x, y\} \in E\}$. For $n \geq 2$ and $L \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, let G$(n, L)$ be the regular graph with vertex set Z_n and edge set E such that $\{x, y\} \in E$ if and only if $|x - y| \in L$.

Theorem 4.1 *For each pair of positive integer n and odd integer s such that* $6n \geq 3s - 1$, *there exists a* 4-*MHF*$_3((2n)^3 : s)$.

Proof For $(n, s) = (1, 1)$, the design exists by Lemma 4.6. For each pair of positive integer n and odd integer s such that $6n \geq 3s - 1$ and $(n, s) \neq (1, 1)$, we construct the auxiliary designs $(D, H, R_0, R_1, R_2)_{(n,s)}$ on Z_{6n}, which are similar to but different from A-pairings in [11]. Here the edge length $2n$ is not allowed to occur in any edge.

1. When $n = 2$ and $s = 1$, let $D = \{4, 10\}$, $H = \{\{1, -1\}, \{2, 7\}\}$, $R_0 = \{\{3, 6\}\}$, $R_1 = \{\{5, 8\}\}$ and $R_2 = \{\{0, 9\}\}$.
2. When $n \geq 3$ and $s = 1$, let $D = \{2n, 4n - 1\}$, $H = \{\{1, -1\}, \{2, -2\}\}$, $R_0 = \{\{0, 2n - 1\}\} \cup \{\{k, 2n-k+1\} : k = 3, 4, \ldots, n\}$, $R_1 = \{\{2n+k, 4n-k-1\} : k = 1, 2, \ldots, n-1\}$ and $R_2 = \{\{4n + k, 6n - 3 - k\} : k = 0, 1, \ldots, n - 2\}$.
3. When odd $s \geq 3$ and $6n \geq 3s - 1$, the $(D, H, R_0, R_1, R_2)_{(n,s)}$ is constructed recursively from the $(D', H', R'_0, R'_1, R'_2)_{(n,s-2)}$. Let r_i be any member of R'_i. Then $D = D' \cup (\cup_{i=0}^{i=2} r_i)$, $H = H'$ and $R_i = R'_i \setminus \{r_i\}$, $i = 0, 1, 2$.

For each pair of (n, s) above, it is easy to check that the complement of the graph $G(6n, LH \cup LR_i \cup \{2n\})$ has a one-factorization $F_i^{(1)}|F_i^{(2)}|\ldots|F_i^{(4n+s-6)}$ for each $i = 0, 1, 2$. Now the desired 4-MHF$_3((2n)^3 : s)$ is constructed on $X = \{a_i : a \in Z_{6n}, i \in Z_3\} \cup \{\infty_1, \infty_2, \ldots, \infty_{3s-1}\}$ with groups $G_{i,j} = \{j_i, (j + 2n)_i, (j + 4n)_i\}$, $i = 0, 1, 2$, $j = 0, \ldots, 2n - 1$ and groups $G_{\infty,j} = \{\infty_j, \infty_{j+s}, \infty_{j+2s}\}$, $j = 1, 2, \ldots, s - 1$ and $G_{\infty,s} = \{\infty_s, \infty_{2s}\}$. And there are three holes $F_i = \{G_{i,j} : j = 0, \ldots, 2n - 1\} \cup S$, $i = 0, 1, 2$ intersecting on the common hole $S = \{G_{\infty,j} : j = 1, 2, \ldots, s\}$. Let the block set be $\mathcal{B}$, which consists of the following five sets of blocks.

$$\begin{aligned}
\delta &= \{\{\infty_j, (a + d)_0, (b - d)_1, (c + d)_2\} : \ a + b + c \equiv 0 \pmod{6n}, \\
&\quad\ d \text{ is the } j\text{th member of } D, 1 \leq j \leq 3s - 1\}, \\
\rho &= \{\{(a + q)_i, (a + t)_i, b_{i+1}, c_{i+2}\} : a + b + c \equiv 0 \pmod{6n}, \\
&\quad \{q, t\} \in R_i, i \in Z_3\}, \\
\phi &= \{\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a, b\} \in F_i^{(k)}, \{c, d\} \in F_{i+1}^{(k)}, \\
&\quad 1 \leq k \leq 4n + s - 6, i \in Z_3\}, \\
\chi_1 &= \{\{a_{i+1}, (a + 3\epsilon)_{i+2}, (x - 2a - 3\epsilon)_i, (y - 2a - 3\epsilon)_i\} : \\
&\quad a \in Z_{6n}, i \in Z_3, \epsilon \in Z_{2n}, \{x, y\} \in H\} \text{ and} \\
\chi_2 &= \{\{a_i, (a + |x - y|)_i, (a + 3\epsilon)_{i+1}, (a + 3\epsilon + |x - y|)_{i+1}\} : \\
&\quad a \in Z_{6n}, i \in Z_3, \epsilon \in Z_{2n}, \{x, y\} \in H\}.
\end{aligned}$$

The rest of the proof is similar to that in [11]. We omit the details here. □

Lemma 4.9 *There exists a good GDC$(3, 4, 3n)$ of type 3^n for all $n \equiv 5 \pmod 8$.*

Proof For $n = 5$, the required design comes from Lemma 4.2. For each given $n \equiv 5, 13 \pmod{24}$ and $n \geq 13$, take a CQS$(1^{(n-1)/4} : 1)$ obtained from an SQS$((n + 3)/4)$. Applying Lemma 2.4 with a 4-MHF$_3(4^3 : 1)$ and a GDD$(3, 4, 48)$ of type 12^4, we get a 4-MHF$_3(4^{(n-1)/4} : 1)$. Then apply Lemma 4.3 with a good GDC$(3, 4, 15)$ of type 3^5 to get the desired design. Here the input ingredient design 4-MHF$_3(4^3 : 1)$ comes from Theorem 4.1.

For $n = 21$, take a 4-MHF$_3(6^3 : 3)$ in Theorem 4.1. Apply Lemma 4.3 with an incomplete good GDC$(3, 4, 27)$ of type $(3^9 : 3^3)$ from Lemma 4.5 and a good GDC$(3, 4, 27)$ of type 3^9 from Lemma 4.2 to get a good GDC$(3, 4, 63)$ of type 3^{21} and an incomplete good GDC$(3, 4, 63)$ of type $(3^{21} : 3^9)$.

For $n = 45$, take a 4-MHF$_3(12^3 : 9)$ in Theorem 4.1. Apply Lemma 4.3 with a good GDC$(3, 4, 63)$ of type 3^{21} and an incomplete good GDC$(3, 4, 63)$ of type $(3^{21} : 3^9)$ to get a good GDC$(3, 4, 135)$ of type 3^{45} and an incomplete good GDC$(3, 4, 135)$ of type $(3^{45} : 3^{21})$.

For $n=69$, take a 4-MHF$_3(8^3 : 1)$ from Theorem 4.1. Apply Lemma 4.3 with a good GDC$(3,4,27)$ of type 3^9 and an incomplete good GDC$(3,4,27)$ of type $(3^9 : 3^3)$ to get an incomplete good GDC$(3,4,75)$ of type $(3^{25} : 3^3)$ and a good GDC$(3,4,75)$ of type 3^{25}. Then take a 4-MHF$_3(22^3 : 3)$ by Theorem 4.1 and apply Lemma 4.3 with an incomplete good GDC$(3,4,75)$ of type $(3^{25} : 3^3)$ and a good GDC$(3,4,75)$ of type 3^{25} to get a good GDC$(3,4,207)$ of type 3^{69}.

For each $n=24k+21$ and $k\geq 3$, we first claim that there exists a CQS$(6^k : 6)$. In fact, assume $(X,\mathcal{G},\mathcal{B})$ is a GDD$(3,4,6(k+1))$ of type 6^{k+1} with $\mathcal{G}=\{G_i : i=1,2,\ldots,k+1\}$. For each $i=1,2,\ldots,k$, there exists a one-factorization $F_i^{(1)}|F_i^{(2)}|\cdots|F_i^{(5)}$ on the complete graph with vertex set G_i. For each pair $\{i,j\}\subset\{1,2,\ldots,k\}$, let $\mathcal{A}_{i,j}=\{\{a,b,c,d\} : \{a,b\}\in F_i^{(l)}, \{c,d\}\in F_j^{(l)}, l=1,2,\ldots,5\}$. Then $\mathcal{B}\cup(\cup_{\{i,j\}\subset\{1,2,\ldots,k\}}\mathcal{A}_{i,j})$ is the block set of a CQS$(6^k : 6)$ on X with groups $\mathcal{G}\setminus\{G_{k+1}\}$ and stem G_{k+1}. Now take this CQS$(6^k : 6)$, apply Lemma 2.4 with a 4-MHF$_3(4^3 : 1)$ and a GDD$(3,4,48)$ of type 12^4. The result is a 4-MHF$_3(24^k : 21)$. Then apply Lemma 4.3 with a good GDC$(3,4,135)$ of type 3^{45} and an incomplete good GDC$(3,4,135)$ of type $(3^{45} : 3^{21})$ to get the desired design. □

Theorem 4.2 *For each pair of positive integer n and odd integer s such that* $6n\geq 3s-1$, *there exists a* 4-*MHF*$_3((2n)^4 : s)$.

Proof For each n and odd s with $6n\geq 3s-1$, Granville and Hartman in [7] constructed a CQS$((6n)^4 : 3s-1)$ on $X=\{a_i : a\in Z_{6n}, i\in Z_4\}\cup\{\infty_1,\infty_2,\ldots,\infty_{3s-1}\}$ with four groups $\{\{a_i : a\in Z_{6n}\} : i\in Z_4\}$ and stem $\{\infty_1,\infty_2,\ldots,\infty_{3s-1}\}$ by defining a Hanani factorization, which is a four-tuple $(D,E,\mathcal{G},\mathcal{H})$ such that $D\subset\{1,3,5,\ldots,6n-1\}$ and $E\subset\{0,2,4,\ldots,6n-2\}$, $|D|=|E|=(3s-1)/2$, $\mathcal{G}=\{G_0,G_1,\ldots,G_{3n-1}\}$ is a set of partial one-factors of the complete graph on vertex set Z_{6n} with $|G_i|=3n-(3s-1)/2$ covering $Z_{6n}\setminus((D\cup E)+2i)$ for $i\in\{0,1,\ldots,3n-1\}$, and $\mathcal{H}$ is a set of one-factors such that $\mathcal{G}\cup\mathcal{H}$ is a partition of the edge set of the complete graph on vertex set Z_{6n}. Now we modify the construction to get a 4-MHF$_3((2n)^4 : s)$. Define Γ to be the graph which covers all the edges in $\mathcal{G}$. By the direct construction of Hanani factorization [7, Theorem 6.1], Γ is cyclic and contains no edge of length $2n$. Let Υ be the complete multipartite graph with $2n$ parts $\{i,i+2n,i+4n\}$, $i=0,1,\ldots,2n-1$. It is not difficult to verify that the complement of Γ in Υ has a one-factorization, which is denoted by $\mathcal{H}'$. Replace $\mathcal{H}$ by $\mathcal{H}'$ in the whole construction of the CQS$((6n)^4 : 3s-1)$. Additionally, replace the one-factorization of the complete graph on Z_{6n}, $J_0|J_1|\ldots|J_{6n-2}$, in the final set of blocks

$$\{\{h_i,\overline{h}_i,a_j,\overline{a}_j\} : \{i,j\}\in\{\{0,1\},\{2,3\}\}, \{h,\overline{h}\}, \{a,\overline{a}\}\in J_k, 0\leq k\leq 6n-2\}$$

with a one-factorization of Υ.

Let $G_{i,j}=\{j_i,(j+2n)_i,(j+4n)_i\}$ for $i=0,1,2,3$, $j=0,\ldots,2n-1$, $G_{\infty,j}=\{\infty_j,\infty_{j+s},\infty_{j+2s}\}$ for $j=1,2,\ldots,s-1$ and $G_{\infty,s}=\{\infty_s,\infty_{2s}\}$. Then, the blocks constructed above will form the block set of a 4-MHF$_3((2n)^4 : s)$ on X with groups $\{G_{i,j} : i=0,1,2,3, j=0,\ldots,2n-1\}\cup\{G_{\infty,j} : j=1,2,\ldots,s\}$ and four holes $F_i=\{G_{i,j} : j=0,\ldots,2n-1\}\cup S$, $i=0,1,2,3$ intersecting on the common hole $S=\{G_{\infty,j} : j=1,2,\ldots,s\}$. □

Lemma 4.10 *There exists a good GDC*$(3,4,3n)$ *of type* 3^n *for all* $n\equiv 1 \pmod 8$ *and* $n\geq 9$.

Proof For each $n=8k+1$ and $k\geq 1$, the proof proceeds by induction. For $k=1$, a good GDC$(3,4,27)$ of type 3^9 exists by Lemma 4.2. When $k>1$, suppose that there exists a good GDC(3, 4, 3(8i + 1)) of type 3^{8i+1} for each $i < k$. By Lemmas 4.8 and 4.9, we have that a good GDC(3, 4, 3j) of type 3^j exists for all odd $j < 8k + 1$. Applying Lemma 4.3 to a 4-MHF$_3((2k)^4 : 1)$ with a good GDC(3, 4, 3(2k + 1)) of type 3^{2k+1}, we get a good GDC(3, 4, 3(8k + 1)) of type 3^{8k+1}. This completes the proof. □

Combining Lemmas 4.8, 4.9 and 4.10, we have

Theorem 4.3 $C(n, 3, 4, 3) = L(n, 3, 4, 3)$ *for all* $n \geq 4$.

5 Optimal constant weight covering codes over Z_{2^m+1}

In this section we focus our attention on the determination of $K_q(n, 4, 3, 1)$ for $n \geq 4$ and $q = 2^m + 1$ with $m \geq 2$. We will give a general result for optimal group divisible coverings with group size 2^m for all $m \geq 2$, i.e., optimal $(n, 4, 3, 1)$ constant weight covering codes over Z_{2^m+1}. From Theorem 2.1, there exists a GDD(3, 4, gn) of type g^n for $g \equiv 2, 4 \pmod 6$, $g \not\equiv 10, 26 \pmod{48}$ and $n \equiv 1, 2 \pmod 3$, which means $C(n, g, 4, 3) = L(n, g, 4, 3)$ for all such pairs of g and n. Now we consider the case for $g \equiv 2, 4 \pmod 6$ and $n \equiv 0 \pmod 3$. It is easy to calculate that $L(n, g, 4, 3) = g^3n(n-1)(n-2)/24 + gn/6$. So if there exists a GDD(3, {4, 6}, gn) of type g^n with exactly $\frac{ng}{6}$ blocks of size 6, then we can get a GDC(3, 4, gn) of type g^n with $L(n, g, 4, 3)$ blocks by replacing the blocks of size 6 with an OGDC(3, 4, 6) of type 1^6.

Similar to the definition in Sect. 3, we call a GDD(3, {4, 6}, gn) of type g^n is *good* if it contains exactly $\frac{ng}{6}$ blocks of size 6.

Lemma 5.1 *If there exists a good GDD*(3, {4, 6}, gn) *of type* g^n, *then there exists a good GDD*(3, {4, 6}, 2gn) *of type* $(2g)^n$.

Proof Suppose that $(X, \mathcal{G}, \mathcal{B})$ is the given good GDD(3, {4, 6}, gn) of type g^n with exactly $gn/6$ blocks of size 6. Let $X' = X \times Z_2$ and $\mathcal{G}' = \{G \times Z_2 : G \in \mathcal{G}\}$. For each $B \in \mathcal{B}$ of size 4, construct a GDD(3, 4, 8) of type 2^4 on $B \times Z_2$ with groups $\{x\} \times Z_2$, $x \in B$. Denote the block set by $\mathcal{A}_B$. For each $B \in \mathcal{B}$ of size 6, construct a good GDD(3, 4, 12) of type 2^6 on $B \times Z_2$ with groups $\{x\} \times Z_2$, $x \in B$. Denote the block set by $\mathcal{C}_B$. Then it is easy to check that $(X', \mathcal{G}', (\cup_{B\in\mathcal{B},|B|=4}\mathcal{A}_B) \cup (\cup_{B\in\mathcal{B},|B|=6}\mathcal{C}_B))$ is a GDD(3, {4, 6}, 2gn) of type $(2g)^n$ with $gn/3$ blocks of size 6. Hence the resultant design is a good GDD(3, {4, 6}, 2gn) of type $(2g)^n$. □

For all integers $m \geq 2$, it is clear that $2^m \equiv 2, 4 \pmod 6$ and $2^m \not\equiv 10, 26 \pmod{48}$. Combining Theorem 2.1, Lemma 5.1 together with Lemmas 3.4 and 3.7, we have the following result.

Theorem 5.1 $C(n, 2^m, 4, 3) = L(n, 2^m, 4, 3)$ *for all* $m \geq 2$ *and* $n \geq 4$.

6 Nonuniform group divisible 3-designs with block size four

In this section, we will employ the construction methods using H-frames to establish several new existence results for GDD(3, 4, 2n + u)s of type 2^nu^1 for $u = 4, 6, 8$. As a consequence of the necessary conditions for GDD(3, 4, gn + u) of type g^nu^1 stated in [18, Theorem 3.1], we have the following lemma.

Lemma 6.1 *If there exists a GDD*$(3, 4, 2n+u)$ *of type* $2^n u^1$ *with* $u = 4$*, then* $n \equiv 1 \pmod 3$ *and* $n \geq 4$*; with* $u = 6$*, then* $n \equiv 1 \pmod 3$ *and* $n \geq 7$*; with* $u = 8$*, then* $n \equiv 0, 1 \pmod 3$ *and* $n \geq 6$.

The following recursive construction for nonuniform group divisible 3-designs was first given in [19].

Lemma 6.2 [19] *Let* mn *be even. If there exists a GDD*$(3, 4, mnr + s + t)$ *of type* $(mn)^r(s+t)^1$ *and a GDD*$(3, 4, mn + s + t)$ *of type* $m^n s^1 t^1$*, then there exists a GDD*$(3, 4, mnr + s + t)$ *of type* $m^{rn} s^1 t^1$.

The following construction is a modification of the filling holes construction for Steiner quadruple systems using candelabra quadruple systems.

Lemma 6.3 *Suppose that there exists a* 4-$HF_g(m^n : s)$*. If there exists a GDD*$(3, 4, g(m+s))$ *of type* $g^{m+\epsilon}(gs - g\epsilon)^1$ *with* $\epsilon = 0$ *or* 1*, then there exists a GDD*$(3, 4, g(mn + s))$ *of type* $g^{mn+\epsilon}(gs - g\epsilon)^1$.

Proof Let $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ be the given 4-HF$_g(m^n : s)$. Let $F_0 = \{G_{\infty,1}, G_{\infty,2}, \ldots, G_{\infty,s}\}$ be the common hole. When $\epsilon = 0$, for each hole $F = \{G_1, G_2, \ldots, G_m\} \cup F_0$ of size $m + s$, construct a GDD$(3, 4, g(m + s))$ of type $g^m(gs)^1$ on $\cup_{G\in F} G$ with group set $\{G_1, G_2, \ldots, G_m\} \cup \{\cup_{G\in F_0} G\}$ and block set $\mathcal{A}_F$. Then $\mathcal{B} \cup (\cup_{F\in\mathcal{F}\setminus F_0} \mathcal{A}_F)$ is the block set of a GDD$(3, 4, g(mn + s))$ of type $g^{mn}(gs)^1$ with group set $\{G \in F \setminus F_0 : F \in \mathcal{F}\} \cup \{\cup_{G\in F_0} G\}$. When $\epsilon = 1$, for each hole $F = \{G_1, G_2, \ldots, G_m\} \cup F_0$ of size $m + s$, construct a GDD$(3, 4, g(m + s))$ of type $g^{m+1}(gs - g)^1$ on $\cup_{G\in F} G$ with group set $\{G_1, G_2, \ldots, G_m, G_{\infty,1}\} \cup \{(\cup_{G\in F_0} G) \setminus G_{\infty,1}\}$ and block set $\mathcal{C}_F$. Then $\mathcal{B} \cup (\cup_{F\in\mathcal{F}\setminus F_0} \mathcal{C}_F)$ is the block set of a GDD$(3, 4, g(mn + s))$ of type $g^{mn+1}(gs - g)^1$ with group set $\{G \in F \setminus F_0 : F \in \mathcal{F}\} \cup \{G_{\infty,1}\} \cup \{(\cup_{G\in F_0} G) \setminus G_{\infty,1}\}$. □

6.1 $u = 4$

Lemma 6.4 [19] *There exists a GDD*$(3, 4, 2n + 4)$ *of type* $2^n 4^1$ *for* $n = 4, 7$.

Theorem 6.1 *There exists a GDD*$(3, 4, 2n + 4)$ *of type* $2^n 4^1$ *if and only if* $n \equiv 1 \pmod 3$ *and* $n \geq 4$.

Proof For each given $n = 3k+1$ and $n \geq 10$, there exists a GDD$(3, 4, 6(k+1))$ of type 6^{k+1} by Theorem 2.1. Applying Lemma 6.2 with $m = 2, n = 3, s = 2, t = 4$ and a GDD$(3, 4, 12)$ of type $2^4 4^1$, we get a GDD$(3, 4, 2n + 4)$ of type $2^n 4^1$. For $n = 4, 7$, the desired designs exist by Lemma 6.4. □

6.2 $u = 6$

Lemma 6.5 *There exists a* 4-$HF_2(3^5 : 1)$.

Proof The desired design is obtained by applying Lemma 2.4 with a CQS$(3^5 : 1)$ in [1] and a GDD$(3, 4, 8)$ of type 2^4. □

Lemma 6.6 *There exists a GDD*$(3, 4, 20)$ *of type* $2^7 6^1$.

Proof We construct the design on Z_{20} with group set $\mathcal{G} = \{\{i, i+7\} : 0 \leq i \leq 6\} \cup \{\{14, 15, 16, 17, 18, 19\}\}$. We list the base blocks below which are developed under the automorphism group

$$G = \langle(0\ 1\ 2)(3\ 4\ 5)(6)(7\ 8\ 9)(10\ 11\ 12)(13)(14\ 15\ 16)(17\ 18\ 19)\rangle.$$

{0, 4, 13, 18}	{1, 2, 7, 15}	{2, 3, 4, 15}	{3, 9, 13, 18}	{1, 9, 13, 17}
{5, 6, 11, 14}	{0, 3, 8, 18}	{4, 7, 10, 15}	{1, 4, 12, 14}	{0, 1, 4, 17}
{0, 2, 5, 14}	{0, 6, 9, 15}	{3, 4, 7, 17}	{3, 11, 12, 18}	{1, 7, 9, 18}
{2, 10, 12, 14}	{0, 4, 6, 19}	{7, 10, 13, 17}	{0, 5, 6, 18}	{0, 3, 5, 17}
{1, 4, 5, 7}	{4, 5, 9, 17}	{4, 6, 7, 18}	{8, 9, 12, 16}	{2, 8, 12, 15}
{1, 4, 10, 13}	{7, 11, 13, 16}	{4, 6, 8, 12}	{0, 9, 12, 18}	{0, 1, 12, 19}
{1, 10, 12, 15}	{3, 7, 8, 14}	{1, 2, 11, 19}	{3, 5, 13, 16}	{0, 1, 5, 11}
{0, 1, 13, 16}	{0, 3, 6, 14}	{8, 10, 11, 14}	{2, 4, 7, 14}	{7, 8, 12, 18}
{3, 7, 12, 19}	{3, 7, 9, 15}	{3, 8, 12, 17}	{0, 6, 8, 11}	{2, 8, 10, 18}
{1, 9, 10, 11}	{5, 7, 8, 10}	{0, 4, 8, 14}	{1, 11, 13, 15}	{5, 6, 7, 15}
{3, 5, 11, 15}	{5, 7, 13, 14}	{1, 3, 11, 17}	{6, 7, 8, 17}	{1, 12, 13, 18}
{6, 7, 12, 14}	{1, 3, 7, 13}	{2, 6, 11, 15}	{3, 11, 13, 19}	{7, 12, 13, 15}
{7, 10, 11, 18}	{0, 6, 10, 17}	{6, 10, 11, 19}	{1, 5, 10, 16}	{0, 1, 2, 6}
{3, 4, 5, 6}	{7, 8, 9, 13}	{10, 11, 12, 13}.		

□

Lemma 6.7 *There exists a GDD*$(3, 4, 32)$ *of type* $2^{13}6^1$.

Proof We construct the design on $Z_{26} \cup \{\infty_0, \ldots, \infty_5\}$ with group set $\mathcal{G} = \{\{i, i+13\} : 0 \leq i \leq 12\} \cup \{\{\infty_0, \ldots, \infty_5\}\}$. We list the base blocks below which are developed under the cyclic group Z_{26}:

$\{0, 15, 19, \infty_0\}$	$\{0, 8, 20, \infty_0\}$	$\{0, 9, 25, \infty_0\}$	$\{0, 3, 24, \infty_0\}$
$\{0, 6, 11, \infty_1\}$	$\{0, 10, 24, \infty_1\}$	$\{0, 17, 18, \infty_1\}$	$\{0, 4, 23, \infty_1\}$
$\{0, 12, 15, \infty_2\}$	$\{0, 1, 18, \infty_2\}$	$\{0, 6, 22, \infty_2\}$	$\{0, 19, 24, \infty_2\}$
$\{0, 20, 25, \infty_3\}$	$\{0, 4, 14, \infty_3\}$	$\{0, 2, 9, \infty_3\}$	$\{0, 15, 23, \infty_3\}$
$\{0, 20, 22, \infty_4\}$	$\{0, 9, 12, \infty_4\}$	$\{0, 1, 8, \infty_4\}$	$\{0, 5, 16, \infty_4\}$
$\{0, 1, 15, \infty_5\}$	$\{0, 17, 21, \infty_5\}$	$\{0, 18, 20, \infty_5\}$	$\{0, 3, 19, \infty_5\}$
{0, 10, 14, 18}	{0, 5, 9, 10}	{0, 2, 7, 14}	{0, 2, 16, 19}
{0, 19, 23, 25}	{0, 22, 23, 24}	{0, 3, 9, 21}	{0, 6, 15, 20}
{0, 11, 22, 25}	{0, 9, 11, 19}	{0, 4, 19, 20}	{0, 3, 5, 23}
{0, 4, 18, 25}	{0, 8, 17, 23}	{0, 8, 10, 19}	{0, 1, 6, 16}.

□

Theorem 6.2 *There exists a GDD*$(3, 4, 2n+6)$ *of type* 2^n6^1 *for all* $n \equiv 1 \pmod 6$ *and* $n \geq 7$.

Proof For $n = 7, 13$, the required designs exist by Lemmas 6.6 and 6.7. For each given $n \equiv 1 \pmod 6$ and $n \geq 19$, there exists a $\{4, 6\}$-CS$(2^{(n-1)/6} : 2)$. Apply Lemma 2.4 with a 4-HF$_2(3^{k-1} : 1)$ and a GDD$(3, 4, 6k)$ of type 6^k with $k \in \{4, 6\}$ to obtain a 4-HF$_2(6^{(n-1)/6} : 4)$. Applying Lemma 6.3 with a GDD$(3, 4, 22)$ of type 2^76^1 in Lemma 6.6, we get a GDD$(3, 4, 2n+6)$ of type 2^n6^1. Here, the small ingredients are from Theorem 3.1 and Lemma 6.5. □

6.3 $u = 8$

Lemma 6.8 [19] *There exists a GDD*$(3, 4, 2n + 8)$ *of type* $2^n 8^1$ *for* $n = 6, 7$.

Lemma 6.9 *There exists a GDD*$(3, 4, 34)$ *of type* $2^{13} 8^1$.

Proof We construct the design on Z_{34} with group set $\mathcal{G} = \{\{i, i + 13\} : 0 \le i \le 12\} \cup \{\{26, 27, 28, 29, 30, 31, 32, 33\}\}$. The base blocks below will be developed under the following automorphism group:

$$\langle (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ \cdots\ 19\ 20\ 21\ 22\ 23\ 24\ 25)(26\ 27)(28\ 29)(30\ 31)(32\ 33) \rangle$$

$\{1, 4, 7, 26\}$	$\{2, 4, 20, 27\}$	$\{8, 11, 16, 23\}$	$\{0, 6, 11, 21\}$	$\{9, 16, 24, 31\}$
$\{1, 3, 21, 32\}$	$\{0, 15, 24, 31\}$	$\{10, 11, 21, 27\}$	$\{15, 18, 25, 31\}$	$\{9, 21, 23, 24\}$
$\{6, 9, 11, 33\}$	$\{11, 18, 20, 26\}$	$\{0, 14, 18, 30\}$	$\{6, 14, 17, 32\}$	$\{7, 9, 11, 18\}$
$\{0, 7, 11, 26\}$	$\{6, 9, 17, 23\}$	$\{3, 6, 22, 24\}$	$\{8, 20, 24, 32\}$	$\{9, 10, 12, 30\}$
$\{10, 21, 25, 29\}$	$\{6, 16, 20, 23\}$	$\{9, 10, 24, 33\}$	$\{5, 7, 13, 29\}$	$\{2, 9, 10, 14\}$
$\{13, 14, 20, 32\}$	$\{0, 12, 17, 27\}$	$\{7, 10, 11, 28\}$	$\{3, 8, 20, 31\}$	$\{10, 13, 19, 28\}$
$\{15, 16, 21, 31\}$	$\{14, 19, 23, 32\}$	$\{3, 7, 8, 27\}$	$\{13, 19, 23, 30\}$	$\{3, 12, 13, 21\}$
$\{0, 8, 20, 26\}$	$\{0, 10, 19, 33\}$	$\{16, 17, 20, 24\}$	$\{0, 1, 20, 25\}$	$\{1, 2, 19, 28\}$
$\{2, 7, 12, 28\}$	$\{1, 2, 11, 17\}$	$\{1, 13, 20, 29\}$	$\{0, 14, 24, 29\}$	$\{0, 2, 6, 12\}$
$\{0, 17, 21, 23\}$.				

□

Lemma 6.10 *There exists a* 4-*HF*$_2(3^5 : 2)$.

Proof We construct the 4-HF$_2(3^5 : 2)$ on $X = Z_{30} \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}$ with group set $\mathcal{G} = \{G_i = \{i, i + 15\} : 0 \le i \le 14\} \cup \{G_{\infty_i} = \{\infty_i, \infty_{i+2}\} : 0 \le i \le 1\}$ and hole set $\mathcal{F} = \{F_i : 0 \le i \le 5\}$ with the common hole $F_0 = \{G_{\infty_i} : 0 \le i \le 1\}$ and $F_i = \{G_{i-1}, G_{i+4}, G_{i+9}\} \cup F_0$ for $1 \le i \le 5$. We list the base blocks below which are developed under the group Z_{30}.

$\{0, 1, 8, \infty_0\}$	$\{0, 2, 13, \infty_0\}$	$\{0, 3, 9, \infty_0\}$	$\{0, 4, 16, \infty_0\}$	$\{0, 4, 6, 20\}$
$\{0, 1, 13, \infty_1\}$	$\{0, 2, 6, \infty_1\}$	$\{0, 3, 11, \infty_1\}$	$\{0, 7, 16, \infty_1\}$	$\{0, 1, 2, 22\}$
$\{0, 1, 3, \infty_2\}$	$\{0, 4, 13, \infty_2\}$	$\{0, 6, 14, \infty_2\}$	$\{0, 7, 18, \infty_2\}$	$\{0, 14, 20, 26\}$
$\{0, 1, 9, \infty_3\}$	$\{0, 2, 14, \infty_3\}$	$\{0, 3, 7, \infty_3\}$	$\{0, 6, 17, \infty_3\}$	$\{0, 3, 13, 20\}$
$\{0, 18, 20, 27\}$	$\{0, 10, 26, 29\}$	$\{0, 3, 17, 25\}$	$\{0, 8, 19, 28\}$	$\{0, 8, 18, 26\}$
$\{0, 19, 26, 27\}$	$\{0, 13, 22, 26\}$	$\{0, 13, 21, 27\}$	$\{0, 12, 14, 23\}$	$\{0, 2, 3, 19\}$
$\{0, 1, 12, 18\}$	$\{0, 2, 24, 27\}$	$\{0, 5, 21, 26\}$	$\{0, 5, 19, 24\}$	$\{0, 13, 18, 25\}$
$\{0, 2, 4, 7\}$	$\{0, 2, 8, 25\}$	$\{0, 18, 21, 28\}$	$\{0, 9, 20, 21\}$	$\{0, 4, 10, 11\}$
$\{0, 1, 7, 14\}$	$\{0, 24, 25, 29\}$.			

□

Lemma 6.11 *There exists a GDD*$(3, 4, 2n+8)$ *of type* $2^n 8^1$ *for all* $n \equiv 0, 1 \pmod 6$, $n \ge 6$ *and* $n \ne 12$.

Proof For $n = 6, 7, 13$, the required designs exist by Lemmas 6.8 and 6.9. For each given $n = 6m + s$, $s \in \{0, 1\}$, $m \ge 3$, there exists a $\{4, 6\}$-CS$(2^{(n-s)/6} : 2)$. Apply Lemma 2.4 with a 4-HF$_2(3^{k-1} : s + 1)$ and a GDD$(3, 4, 6k)$ of type 6^k with $k \in \{4, 6\}$ to obtain a 4-HF$_2(6^{(n-s)/6} : s+4)$. Then applying Lemma 6.3 with a GDD$(3, 4, 2s+20)$ of type $2^{6+s}8^1$, we get a GDD$(3, 4, 2n + 8)$ of type $2^n 8^1$. Here, the input designs are from Theorem 3.1, Lemmas 6.5, 6.8 and 6.10. □

Theorem 6.3 *There exists a* 4-*HF*$_2((3n+s)^3 : s)$ *for all* $n \geq 0$ *and* $s \geq 1$.

Proof In [10, Theorem 3.4], Hartman constructed a CQS$((6n+2s)^3 : 2s)$ for each integer $n \geq 0$ and $s \geq 1$ on $X = \{a_i : a \in Z_{6n+2s}, i \in Z_3\} \cup \{\infty_1, \infty_2, \ldots, \infty_{2s}\}$ with three groups $\{\{a_i : a \in Z_{6n+2s}\} : i \in Z_3\}$ and a stem $\{\infty_1, \infty_2, \ldots, \infty_{2s}\}$. Let the block set be $\mathcal{B}$, in which there is a set of blocks

$$\begin{aligned}\phi = \{\{a_i, b_i, c_{i+1}, d_{i+1}\} &: \{a, b\} \in F_i^{(k)}, \{c, d\} \in F_{i+1}^{(k)},\\ &1 \leq k \leq 4n+2s-1, i \in Z_3\},\end{aligned}$$

where $F_i^{(1)}|F_i^{(2)}|\ldots|F_i^{(4n+2s-1)}$ is a one-factorization of the graph on $Z_{6n+2s} \times \{i\}$ defined by the *simple pairing* $P(n, 2s)$ constructed in [10, Theorem 3.3]. Clearly, $4n + 2s - 1 \geq 1$.

The desired 4-HF$_2((3n+s)^3 : s)$ will be on X with the group set $\mathcal{G} = \{\{a_i, b_i\} : \{a, b\} \in F_i^{(1)}, i \in Z_3\} \cup \{\{\infty_i, \infty_{i+s}\} : 1 \leq i \leq s\}$, three holes $\mathcal{F}_{i+1} = \{\{a_i, b_i\} : \{a, b\} \in F_i^{(1)}\} \cup \mathcal{F}_0$, $i \in Z_3$ intersecting on a common hole $\mathcal{F}_0 = \{\{\infty_i, \infty_{i+s}\} : 1 \leq i \leq s\}$.

Let

$$\phi_1 = \{\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a, b\} \in F_i^{(1)}, \{c, d\} \in F_{i+1}^{(1)}, i \in Z_3\}.$$

Note that $\phi_1 \subset \phi$ and each block in ϕ_1 intersects two groups in $\mathcal{G}$ which are from two distinct holes. It is readily checked that $\mathcal{B} \setminus \phi_1$ is the block set of the desired 4-HF$_2((3n+s)^3 : s)$. □

Lemma 6.12 *There exists a GDD*$(3, 4, 2n+8)$ *of type* $2^n 8^1$ *for all* $n \equiv 3, 16 \pmod{18}$, $n \geq 16$ *and* $n \neq 34$.

Proof For each given $n = 18k + 3$ with $k \geq 1$, there is a 4-HF$_2((3(2k-1)+4)^3 : 4)$ by Theorem 6.3. By applying Lemma 6.3 with a GDD$(3, 4, 12k+10)$ of type $2^{6k+1}8^1$ from Lemma 6.11, we get a GDD$(3, 4, 2n+8)$ of type $2^n 8^1$.

For each given $n = 18k + 16$ with $k \geq 0$ and $k \neq 1$, there is a 4-HF$_2((6k+5)^3 : 5)$ by Theorem 6.3. Then applying Lemma 6.3 with a GDD$(3, 4, 12k+20)$ of type $2^{6k+6}8^1$ from Lemma 6.11, we get a GDD$(3, 4, 2n+8)$ of type $2^n 8^1$. □

Combining Lemmas 6.11 and 6.12, we obtain

Theorem 6.4 *There exists a GDD*$(3, 4, 2n+8)$ *of type* $2^n 8^1$ *for all* $n \equiv 0, 1, 3, 6, 7, 12, 13, 16 \pmod{18}$, $n \geq 6$ *except possibly for* $n = 12, 34$.

7 Conclusion

In this article, we determine the minimum size of a constant weight covering code $(n, 4, 3, 1)$ over Z_q for all $n \geq 4, q = 3, 4$ or $q = 2^m+1$ with $m \geq 2$, leaving the only case $(q, n) = (3, 5)$ in doubt. The problem was solved by establishing an equivalent existence result for group divisible coverings of triples by quadruples using different types of H-frames, which play a crucial role in the recursive constructions of group divisible 3-designs similar to that of candelabra systems in the constructions of 3-wise balanced designs. This approach has also been proved to be quite effective to deal with the existence problems for nonuniform group divisible 3-designs with block size four and types $2^n u^1$ with $u = 6, 8$. We believe that the theory of candelabra systems and H-frames will be proved useful for solving the general existence problem on nonuniform group divisible 3-designs with block size four and type $g^n u^1$.

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